Physics 53

Energy 1

What I tell you three times is true.
— Lewis Carroll

The interplay of mathematics and physics

The "mathematization" of physics in ancient times is attributed to the Pythagoreans, who taught that everything true is contained in numbers. But the introduction of algebraic equations as a way of stating laws of nature dates from the time of Galileo. In Isaac Newton science had both a great innovator in mathematics and a great analyst and experimenter on natural phenomena. Because he was (independently of Leibniz) the discoverer of calculus, he was able to think of physical processes in terms of rates, infinitesimal changes, and summations of infinitesimals into finite quantities by means of integrals. Although he published relatively little of his thinking on these matters — his famous book, the *Principia*, contains no reference to calculus, giving geometrical arguments and proofs instead — it seems clear that he thought analytically, in the modern mathematical sense of that word.

In the 18$^{\text{th}}$ century great advances were made in mathematical analysis, and many of these were applied to — indeed, arose from — problems in physics. New mathematical formulations were found for the content of Newton's three laws of motion, making easier the solutions of many physical problems. The work of Euler and Lagrange in particular gave important new insights, and led to the emergence in the 19$^{\text{th}}$ century of what is probably the most important single concept in science, that of **energy**.

In this part of the course we deal with energy as it applies to the mechanics of particles and systems of particles. Later we extend the concept to fluids and thermal systems, and in the next course we discuss the energy associated with electromagnetic fields.

The scalar product

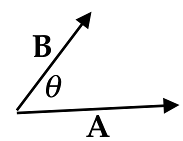
In our discussion of energy we will need to use the product of two vectors that results in a scalar. Consider two vectors, $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$. Multiplying one component of $\mathbf{A}$ by one component of $\mathbf{B}$ gives a set of 9 pairs. This set as a whole is not very useful because it does not transform simply when we rotate the coordinate axes.

Some combinations of the 9 do have simple transformation properties, however. The simplest of these is $A_x B_x + A_y B_y + A_z B_z$. It can be shown to be *unchanged* by rotation of the axes, so it is a scalar, and is therefore called the **scalar product**. The standard notation for it uses the "dot" multiplication sign between the two vectors:

Scalar product	$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$

Because of the notation, it is often called the "dot" product.

We can rewrite this formula in terms of the magnitudes and relative direction of the two vectors. Let the vectors be as shown in arrow representation. Then it is easy to derive a very useful formula:



Scalar product formula	$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$

Some properties of the scalar product $\mathbf{A} \cdot \mathbf{B}$ that follow from this:

- $\mathbf{A} \cdot \mathbf{B}$ is positive if $\theta < \pi / 2$, negative if $\theta > \pi / 2$, zero if $\theta = \pi / 2$.
- When $\theta = 0$, $\mathbf{A} \cdot \mathbf{B}$ has its maximum value $+AB$.
- When $\theta = \pi$, $\mathbf{A} \cdot \mathbf{B}$ has its minimum value $-AB$.
- The scalar product of a vector with itself is its squared magnitude : $\mathbf{A} \cdot \mathbf{A} = A^2$.

A useful way to think of the value of the scalar product is this: it is the magnitude of one vector multiplied by the component of the second vector along the line of the first.

Power, work and kinetic energy

Newtonian analysis of the mechanics of a particle consists of this process:

1. Given the initial conditions of the particle ($\mathbf{r}_0$ and $\mathbf{v}_0$);
2. Given the net interaction it has with its environment ($\mathbf{F}_{\text{tot}}$);
3. Determine its position $\mathbf{r}$ as a function of time.

If we know $\mathbf{F}_{\text{tot}}$ as a function of *time*, Newton's 2nd law gives us $\mathbf{a}(t)$ from which (by two integrations and using $\mathbf{r}_0$ and $\mathbf{v}_0$) we can find $\mathbf{r}(t)$. This is what we have done in the simple cases where $\mathbf{a}$ is a constant, and in a few other cases.

Unfortunately, often we do not know the forces as functions of *time*. More often they are known as functions of the particle's *position*, **r**. From the above argument it seems that to determine **r** we must know **r** in advance, which makes the direct approach hopeless. In these cases we use a trick devised by 18th century scientists, notably Euler.

We start with the one-dimensional case. Assume we know the total force $F(x)$ as a function of the particle's position x. The 2nd law then reads $F(x) = ma$. Now construct the quantity $P = Fv$, where v is the velocity (not the speed, since v can be negative). Then $P = mav$. But since

$$\frac{d}{dt}\left(\tfrac{1}{2}v^2\right) = v\frac{dv}{dt} = va$$

we can also write $P = mav$ as

$$P = \frac{d}{dt}\left(\tfrac{1}{2}mv^2\right).$$

This equation relates two new quantities that turn out to be useful enough to have names: the **power input** by a force

Power input by a force (1-D)	$P = Fv$

and the **kinetic energy** of the particle:

Kinetic energy of a particle	$K = \tfrac{1}{2}mv^2$

What has been shown (in one dimension) is that $P = dK/dt$.

The power input by the total force is equal to the rate of change of the kinetic energy.

All we have really shown is that two quantities we have *defined* are related. It remains to be shown why this is useful. And it remains to be shown that it is true in 3-D.

To find the net change in kinetic energy between two times we integrate over time:

$$K(2) - K(1) = \int_{t_1}^{t_2} P\,dt = \int_{t_1}^{t_2} Fv\,dt = \int_{x_1}^{x_2} F\,dx\,.$$

(We have used the fact that the distance dx moved in time dt is given by $dx = v\,dt$.)

Since we know F as a function of x, in principle we can evaluate the last integral. It also has a name: the **work done by the force** as the particle moves from x_1 to x_2:

Work done by a force (1-D)	$W(1 \to 2) = \int_{x_1}^{x_2} F\, dx$

What we have proved is an important theorem:

Work-energy theorem	The work done by the total force is equal to the change in kinetic energy.

Through this theorem we can find how the particle's kinetic energy (and from that its speed) depends on its position. This approach tells us the speed when the particle is at a particular *place*. It does *not* tell us the *time* when the particle is at that place. It also tells us only the speed, not the *direction* of the velocity. We have not found a complete description of the motion, but it is nevertheless very useful.

Since the change in K is equal to the total work done, and the rate at which K changes is the total power input, it follows that the relation between power and work is

$$P = dW / dt\,.$$

Power input by a force is the rate at which work is done by that force.

Work and kinetic energy have the same dimensions, and in SI units they are both measured in joules, where 1 J = 1 N-m. Power is measured in watts, where 1 W = 1 J/s.

Although work and kinetic energy have the same dimensions, they are different in an important way:

- Kinetic energy is *a property of the particle and its motion*; it is a variable that describes the *state* of the particle. We say that the particle *has* a certain kinetic energy.
- Work is an *effect* of the forces acting on the particle; it refers to a *process* by which the kinetic energy is being changed. It refers to *energy in transit* between the particle and its environment, brought about by the action of the force. The particle does not *have* a certain amount of work; a certain amount of work is *done on* the particle.

A change in the state can come about by different processes in which work is done in different ways. But the *total* work done will *always* equal the change in kinetic energy, regardless of the process used.

Consider, for example, a particle that starts at one place at rest, moves about, and comes to rest at another place. Since the initial and final kinetic energies are both zero, the *total* work done by *all* forces is zero, no matter how complicated the actual motion might be.

Power, work and kinetic energy in 3-dimensions

The trick being used here seems to work nicely in one-dimension, for forces that depend only on x. Does it work also in 3-D, where force, velocity and acceleration are vectors?

Since kinetic energy depends only on the particle's mass and *speed*, it is a scalar quantity. If its relationships to work and power are to be the same in 3-D, then work and power must also be scalars. We are thus led to the general definition of power:

Power input by a force (general)	$P = \mathbf{F} \cdot \mathbf{v}$

We can show that the power input by the total force is still the rate of change of kinetic energy. But we see that the *relative directions* of **F** and **v** must be considered carefully. If the angle between **F** and **v** is less than $\pi/2$, the power input is positive, and the effect of the force is to increase K; if the angle is greater than $\pi/2$ the power input is negative and the force decreases K. If the angle is exactly $\pi/2$ then the power input is zero and the kinetic energy does not change.

How do we define work in 3-D? Consider an infinitesimal time dt during which the particle moves an infinitesimal amount $d\mathbf{r} = \mathbf{v}\,dt$. We look at the quantity

$$dK = \mathbf{F} \cdot \mathbf{v}\,dt = \mathbf{F} \cdot d\mathbf{r} .$$

This shows that the infinitesimal change in K is given by the scalar product of the (total) force **F** and the infinitesimal change in position $d\mathbf{r}$. This product must be the (infinitesimal) work done by the total force:

$$dW = \mathbf{F} \cdot d\mathbf{r} .$$

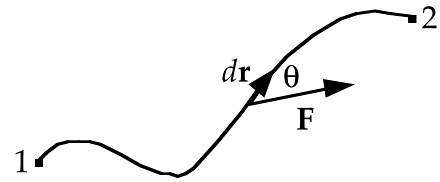
To get the total work done by any force **F** as the particle moves from point 1 to point 2 we integrate this quantity along the path followed between these points:

Work done by a force (general)	$W(1 \to 2) = \int_1^2 \mathbf{F} \cdot d\mathbf{r}$

With this definition of work, the work-energy theorem is also valid in three dimensions.

The integral defining work is taken along the path actually followed by the particle; it is called a **line integral**. In a sense it is a one-dimensional integral, but it is more complicated than the usual cases studied in calculus courses, because the path need not be a straight line.

To understand its meaning and how it might be calculated, consider the endpoints (1 and 2) and the trajectory followed by the particle. The infinitesimal displacement $d\mathbf{r}$ is tangent to the curve, and we see from the drawing that $dW = F\,dr\cos\theta$, where dr is the magnitude of $d\mathbf{r}$. This gives us the work done in that infinitesimal part of the path. The total work integral is the sum of these infinitesimal contributions as the particle moves from point 1 to point 2.



The value of this kind of integral generally depends not only on the end points (1 and 2) and the force but *also on the path followed*. If it does depend on the path, then the work done (and thus the change in kinetic energy) is different for different paths, and this approach to analyzing the motion becomes more complicated.

However, there are important cases where the work is either simply *zero*, or else *independent* of the path:

1. Suppose the particle is sliding on a *stationary* solid surface. The normal force **N** exerted by the surface is (by definition) perpendicular to that surface, while the velocity **v** is parallel to it, so the angle θ between **N** and **v** is always $\pi/2$ and $\mathbf{N}\cdot\mathbf{v} = 0$. The power input by the normal force exerted by a stationary surface is thus *always zero*, no matter what path is followed, so it does not change the kinetic energy (or speed) of the particle.

   If the solid surface is not stationary, the particle's velocity is not necessarily along the surface, so the normal force can change the kinetic energy.

2. The radial force in circular motion, being perpendicular to the trajectory, does no work and does not change the kinetic energy.

3. Suppose the force **F** under consideration is *constant* in both magnitude and direction. Then it can be moved outside the work integral:

$$W(1 \to 2) = \mathbf{F}\cdot\int_1^2 d\mathbf{r} = \mathbf{F}\cdot(\mathbf{r}_2 - \mathbf{r}_1).$$

   In this case the work depends *only* on the endpoints and *not* on the path.

As we will see, there are other important cases where the work done is independent of the path followed by the particle. Forces with this property are called *conservative*.

A familiar case where the work done *depends* on the path is kinetic friction. Even if the magnitude of this force remains constant, its direction is always *opposite* to the motion, so as the particle's motion changes direction the friction force changes direction accordingly. *Kinetic friction is not a constant force*, even if the motion is one-dimensional. In fact, since the friction force is always opposite to the particle's velocity, work done by kinetic friction is *always* negative.

Suppose the work by the total force *is* independent of the path. Then if the starting and ending points are the same the total work done must be *zero*. (One "path" to go from a point to itself is not to go at all.) This means that the particle's speed is the same at the end as it was at the beginning. During part of the travel the force may have increased the speed; if so, then during the other part it decreased it by the same amount.

Although the *speed* is the same when the particle returns to the starting point, the *direction* of its velocity might be different. An example is the point at the bottom of the swing of a pendulum. Each time the particle passes through that point its speed is the same, but the direction of the velocity alternates.